A two-nonlinearity model of dissipative drift wave turbulence

Y.-M. Liang, P. H. Diamond,[a] and X.-H. Wang[b]
Department of Physics, University of California, San Diego, California 92093

D. E. Newman and P. W. Terry
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706

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A simple, one-field, two-nonlinearity, drift wave model equation is derived to describe the dynamics of a nonuniform magnetized plasma by taking into account the effects of dissipative trapped electron response in the turbulence dynamics. Because of the nonadiabatic response of trapped electrons, mode couplings by both the $\mathbf{E}\times\mathbf{B}$ drift and the polarization drift nonlinearities are present. In this work, the statistical dynamics for this dissipative drift wave turbulence is investigated using the EDQNM (eddy-damped quasinormal Markovian) closure scheme. In particular, apart from the eddy viscosity, a large nonlinear frequency shift is shown to be induced by cross coupling of the two nonlinearities. Thus instability drive is modified by this turbulent back reaction. By taking into account this self-consistency effect, a wave kinetic equation is derived, and the density fluctuation spectrum is obtained in different parameter ranges. The results show that the dynamics of dissipative drift wave turbulence is fundamentally different from that of the familiar Hasegawa–Mima model, because $\mathbf{E}\times\mathbf{B}$ drift nonlinearity blocks the low-k condensation of fluctuation energy. It is shown that both the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity and the nonlinear frequency shift effect transfer energy nonlocally from large to small scales and, in contrast to the predictions of dimensional analysis, their contribution to the nonlinear transfer processes are actually of the same order as that of the polarization drift nonlinearity, *even* within the Hasegawa–Mima regime. This results in a significant modification of the Hasegawa–Mima spectrum for the short-wavelength drift waves.

I. INTRODUCTION

Drift waves have frequently been associated with the observed low-frequency density fluctuations and energy confinement degradation in tokamaks. Thus drift wave turbulence has been of considerable interest in plasma fusion research. Despite many studies,[1–4] the basic dynamics is still rather poorly understood. Most previous studies of this subject have utilized a simple, one-field, nonlinear fluid model for the potential fluctuation $\tilde{\phi}$ known as the Hasegawa–Mima equation.[1] In its derivation, the electron response is assumed to be adiabatic so that $\tilde{n}=\tilde{\phi}$, where $\tilde{n}$ and $\tilde{\phi}$ are the normalized density and potential fluctuations, respectively. Because of the adiabatic electron approximation, only the polarization drift nonlinearity appears in the Hasegawa–Mima equation. This simple model equation is similar to the two-dimensional Navier–Stokes equation and admits two inviscid invariants of motion, i.e., the total energy and the total enstrophy. Therefore, based on intuition from two-dimensional hydrodynamic turbulence, the Hasegawa–Mima model of drift wave turbulence predicts an inverse cascade of total energy,[2] from large $k_\perp$ to small $k_\perp$. However, since it completely ignores density fluctuation dynamics by taking the electrons to be adiabatic, the Hasegawa–Mima equation is not a general model for drift wave turbulence, and conclusions reached based on it are not universal. Later on, "$i\delta$" models[3,4] were introduced to simulate the nonadiabatic electron response by adding an "$i\delta$" term to the electron density response function, i.e., $\tilde{n}_{\mathbf{k}}=\tilde{\phi}_{\mathbf{k}}(1-i\delta_{\mathbf{k}})$. In the "$i\delta$" models, in addition to the polarization drift nonlinearity, the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, induced by the $\mathbf{E}\times\mathbf{B}$ convection of the nonadiabatic electrons, also appears in the basic equation. It has been shown in numerical simulations that this $\mathbf{E}\times\mathbf{B}$ drift nonlinearity plays an important role in mode coupling processes.[3] However, most of the works[3,4] using "$i\delta$" models are not self-consistent, in the sense that the effects of turbulent back reaction on the instability drive, because of the cross-coupling effect of the two nonlinearities, are totally ignored. Moreover, the dynamical interaction of the two nonlinearities were not properly treated. Conventional treatments analyze each class of scales individually, ignoring nonlocal interactions between the small and large scales. In particular, when considering the nonlinear dynamics of the short-wavelength drift waves (i.e., in the Hasegawa–Mima regime, where $k_\perp \rho_s \sim 1$), the limit of $\delta_{\mathbf{k}}=0$ was naively taken by completely neglecting the random modulational effect from large scale (i.e., $k_\perp \rho_s \ll 1$) fluctuations. Also, most previous works on the "$i\delta$" models are numerical simulations,[3,4] and little analytical theory and understanding of these models is available. In a recent study by Gang *et al.*[5] on a two-field model[6] of dissipative drift wave turbulence, the statistical dynamics was investigated and closure equations were derived. However, the complexity of the renormalized closure equations derived for this two-field model prevented the authors from analytically obtaining the saturation spectrum, etc. On the

[a]Also at General Atomics, San Diego, California 92186-9784.
[b]Present address: Columbia University, New York, New York 10027.

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other hand, recent works on very-long-wavelength drift wave turbulence,[7,8] which is destabilized by dissipative trapped particle instabilities, revealed some novel features of drift wave turbulence. In particular, Diamond and Biglari[7] showed that no long-wavelength condensation of fluctuation energy is possible in the dissipative trapped-ion convective-cell turbulence, where the nonlinear mode coupling process is through the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity effected by the $\mathbf{E}\times\mathbf{B}$ convection of the nonadiabatic trapped ions. This low-k condensation blocking effect due to the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity was confirmed in simulations by Newman *et al.*[8]

Obviously, the inhibition of long-wavelength condensation of fluctuation energy, effected by the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, will compete with the inverse cascade of fluctuation energy, effected by the polarization drift nonlinearity. With the above observations, we are naturally prompted to construct a simple, one-field model of dissipative drift wave turbulence, which will properly incorporate the nonadiabatic trapped particle response into the ion dynamics. In a toroidal geometry such as in tokamaks, drift modes fall into two branches. One is the usual slab-type Pearlstein–Berk mode which oscillates both radially and along the magnetic field lines, and is damped by ion Landau resonance. The other branch, the so-called toroidicity-induced mode, is slowly varying along the magnetic field lines, i.e., $k_{\parallel} \ll k_{\perp}$, and is localized radially. Since the toroidicity-induced modes are quasibounded, they experience minimal magnetic shear damping, hence, they are easily destabilized. Also, apart from the details of coupling, they are fairly well modeled by local theory. For this reason, a study of the local analog of these toroidicity-induced-like modes is more relevant. Therefore we consider a shearless magnetic field model. (The investigation of this model in a sheared magnetic field will be presented in a future publication.) We use a cool fluid model for the ions, and start from the ion continuity equation, in which the variation of the drift modes along the magnetic field lines is neglected. As for the dynamics of dissipative trapped particles, in order for us to construct a simple, one-field equation, we consider the trapped electron instability, and assume that the trapped electrons are *strongly* dissipative (i.e., $\nu_{\text{eff}} \gg \omega_{\mathbf{k}}$). Finally, the system of equations is closed with the quasineutrality condition.

Before we systematically investigate the dynamics of the above model, we can intuitively anticipate some of the results. Because of the nonadiabatic response of the trapped electrons, the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity exists, in addition to the familiar polarization drift nonlinearity encountered in the Hasegawa–Mima equation. However, the nonlinear mode-coupling processes mediated by these two nonlinearities are distinctly different, since the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity alone tends to transfer energy to small scales,[7,8] while the polarization drift nonlinearity alone transfers energy to large scales.[2] In the case when both the nonlinearities coexist, the breakdown of enstrophy conservation by the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity eliminates the familiar dual-cascade mechanism. Thus the usual nonlinear transfer picture and saturation spectrum for drift wave turbulence based on the Hasegawa–Mima model is expected to be modified by the interplay of both nonlinearities.

Indeed, according to our analysis, we show that the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity blocks the low-k condensation of fluctuation energy from the polarization drift nonlinearity. Even in the short-wavelength regime, where the polarization drift nonlinearity is predicted to dominate by previous dimensional analysis,[4] the $\mathbf{E}\times\mathbf{B}$ nonlinearity plays a significant role. Here, we enumerate the major results of this work:

(1) We show that the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity transfers energy *nonlocally* from large to small scales, and that the polarization drift nonlinearity transfers energy *locally* from small to large scales. Because of this nonlocal versus local nature of the energy transfer process, the effect of the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity is shown to be of the same order as that of the polarization drift nonlinearity, even in the short-wavelength regime. Thus approximating $\delta_{\mathbf{k}}=0$ in this regime is inappropriate.

(2) A nonlinear frequency shift is found to be induced by the cross coupling of the $\mathbf{E}\times\mathbf{B}$ drift and polarization drift nonlinearities, and it is of the order of the electron diamagnetic drift frequency. This nonlinear frequency shift effectively supplies another "nonlinearity" by self-consistently modifying the instability drive. It is shown that the energy "flow" mediated by this frequency shift effect is from large to small scales, as is the transfer due to the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity.

(3) At large scales where $k_{\perp}\rho_s \ll 1$, it is shown that the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity dominates, and the fluctuation spectrum is determined by balancing it against the linear instability drive. The spectrum is shown to be rather flat over most of this regime. However, it decreases quickly to low levels near the crossover point $k_{\perp}\rho_s=\xi$. The flatness of the fluctuation spectrum is due to the dominance of the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, which has only one inviscidly conserved quantity, the energy. According to the predictions of closure theory, the nonlocal transfer mediated leads to a flat spectrum.

(4) At small scales where $k_{\perp}\rho_s \sim 1$, however, all the nonlinear transfer processes, including the nonlinear frequency shift effect, have to be accounted for, even though the polarization drift nonlinearity is predicted to be dominant from naive dimensional analysis. This leads to a *large* modification of the Hasegawa–Mima spectrum, and the inverse energy transfer process by the polarization drift nonlinearity is greatly altered by the low-k condensation blocking effect of the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity.

The remainder of this paper is organized as follows. In Sec. II, the basic model equation is derived, and the properties of the two nonlinearities are briefly discussed. In Sec. III, the EDQNM closure method is applied to the nonlinear model equation, and the nonlinear dispersion relation is obtained from one-point renormalization, where apart from the eddy viscosity terms, a nonlinear frequency shift term is found to be induced by the cross coupling of the two nonlinearities. The dynamic impact of this nonlinear frequency shift on the instability drive is discussed in Sec. IV. By including this self-consistent turbulent back reaction, the wave kinetic equation for the density fluctuation spectrum is derived in Sec. V, and in Sec. VI, the stationary density fluctuation spectra are obtained in different spectral regimes. Finally, we summarize and discuss our results in Sec. VII.

II. MODEL EQUATION

In this section, we derive a single-equation, dual-nonlinearity drift wave model. In a typical linear Ohmic confinement regime tokamak, the electron temperature is higher than the ion temperature, and we use a cool fluid approximation for the ions. In the derivation, the underlying instability of the system is the strongly dissipative trapped electron mode, which supplies an $i\delta$-type instability drive. The strong dissipation allows the entire time evolution to be combined into a single nonlinear equation.

We use a standard shearless slab geometry, in which x refers to the radial coordinate, y to the poloidal coordinate, and z to the toroidal coordinate, and the equilibrium quantities are functions of x only. The equilibrium magnetic field is $\mathbf{B}=B_0\hat{z}$. In addition to the magnetic field, the equilibrium is characterized by electron density and temperature profiles with basic scale lengths L_n and L_T, respectively. In the present model, *no* equilibrium radial electric field is assumed.

As for the electron dynamics, we focus our analysis on the dissipative trapped electron regime,[9] where the trapped electron response satisfies

$$-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}}^{De}+i\nu_{\text{eff}})\widetilde{g}_{\mathbf{k}}$$
$$=i\frac{|e|\widetilde{\phi}_{\mathbf{k}}}{T_{e0}}\sqrt{\epsilon}f_0\{\omega_{\mathbf{k}}-\omega_{*\mathbf{k}}[1+\eta_e(v^2/v_e^2-3/2)]\}. \quad (1)$$

Here, ν_{eff} is the effective collision frequency, $\omega_{\mathbf{k}}^{De}$ is the electron curvature drift frequency, $\widetilde{g}_{\mathbf{k}}$ is the component of the nonadiabatic part of the perturbed distribution function with wave frequency $\mathbf{k}$, $\sqrt{\epsilon}$ is the fraction of trapped electrons, f_0 is the equilibrium distribution function, $\omega_{*\mathbf{k}}=k_yV_{*n}=k_yc_s\rho_s/L_n$ is the electron diamagnetic drift frequency, $c_s=\sqrt{T_{e0}/m_i}$ is the sound velocity, $\rho_s=c_s/\Omega_i$ is the ion gyroradius, v_e is the electron thermal velocity, and $\eta_e\equiv L_n/L_T$. For strongly dissipative trapped electron modes, $\nu_{\text{eff}}\gg\omega_{\mathbf{k}},\omega_{\mathbf{k}}^{De}$. After taking this limit in Eq. (1) and integrating over velocity space, we obtain the nonadiabatic electron density response

$$\frac{\delta n_{\mathbf{k}}^{\text{NA}}}{n_0}=i\frac{|e|\widetilde{\phi}_{\mathbf{k}}}{T_{e0}}\sqrt{\epsilon}\frac{\omega_{\mathbf{k}}-\omega_{*\mathbf{k}}(1+\alpha\eta_e)}{\nu_{\text{eff}}}, \quad (2)$$

where $\alpha=3/2$. Normally, a standard drift wave turbulence result predicts that $\omega_{\mathbf{k}}\simeq\omega_{*\mathbf{k}}/(1+k_\perp^2\rho_s^2)$. However, in general, there may be a *nonlinear frequency shift* $\omega_{\mathbf{k}}^s$ away from $\omega_{*\mathbf{k}}/(1+k_\perp^2\rho_s^2)$, thus we write

$$\omega_{\mathbf{k}}=\frac{\omega_{*\mathbf{k}}}{1+k_\perp^2\rho_s^2}+\omega_{\mathbf{k}}^s. \quad (3)$$

Since we order $\eta_e>1$, for simplicity, $(\omega_{\mathbf{k}}-\omega_{\mathbf{k}}^s)-\omega_{*\mathbf{k}}$ is negligible compared to $\alpha\eta_e\omega_{*\mathbf{k}}$. Then the nonadiabatic electron density response is given by

$$\frac{\delta n_{\mathbf{k}}^{\text{NA}}}{n_0}\approx i\frac{|e|\widetilde{\phi}_{\mathbf{k}}}{T_{e0}}\sqrt{\epsilon}\frac{-\alpha\eta_e\omega_{*\mathbf{k}}+\omega_{\mathbf{k}}^s}{\nu_{\text{eff}}}, \quad (4)$$

and thus the quasineutrality condition leads to

$$\frac{|e|\widetilde{\phi}_{\mathbf{k}}}{T_{e0}}=\left(1+\alpha\sqrt{\epsilon}\frac{V_{*T}}{\nu_{\text{eff}}}ik_y\right)\frac{\widetilde{n}_{i\mathbf{k}}}{n_0}-i\frac{\sqrt{\epsilon}}{\nu_{\text{eff}}}\omega_{\mathbf{k}}^s\frac{\widetilde{n}_{i\mathbf{k}}}{n_0}. \quad (5)$$

In Sec. III, we will show that the cross coupling of the two nonlinearities supplies a nonlinear frequency shift that is comparable to $\omega_{*\mathbf{k}}$, and that the dynamic impact of this frequency shift effect will be discussed in detail in Sec. IV. For now, we simply consider the case with $\omega_{\mathbf{k}}^s=0$. We will come back to discuss the $\omega_{\mathbf{k}}^s\neq0$ case self-consistently in Sec. IV.

We treat ions as a fluid, and use the continuity equation,

$$\frac{\partial\widetilde{n}_i}{\partial t}+\widetilde{V}_x\frac{dn_0}{dx}+\widetilde{\mathbf{V}}\cdot\nabla\widetilde{n}_i=-n_0(\nabla_\perp\cdot\widetilde{\mathbf{V}}_\perp+\nabla_\parallel\widetilde{V}_\parallel), \quad (6)$$

where the perpendicular ion flow velocity is due to $\mathbf{E}\times\mathbf{B}$ and polarization drift flows. Again, variation of the drift modes along the magnetic field line is neglected, i.e., $\nabla_\parallel=0$. Therefore the model becomes a quasi-two-dimensional problem. Then, in the case when $\omega_{\mathbf{k}}^s=0$, the model equation is derived and is written as

$$\frac{\partial}{\partial t}(1-\rho_s^2\nabla_\perp^2)n+V_{*n}\frac{\partial n}{\partial y}+D_0\frac{\partial^2 n}{\partial y^2}$$
$$-L_nD_0\left[\nabla_\perp\left(\frac{\partial n}{\partial y}\right)\times\hat{z}\right]\cdot\nabla_\perp n$$
$$+\rho_sc_s(\nabla_\perp n\times\hat{z})\cdot\nabla_\perp(\rho_s^2\nabla_\perp^2 n)=0, \quad (7)$$

where the normalized ion density perturbation is $n=\widetilde{n}_i/n_0$, the ion diamagnetic drift velocity is $V_{*n}=c_s\rho_s/L_n$ and $D_0=\alpha\sqrt{\epsilon}(\rho_sc_s)^2/(L_TL_n\nu_{\text{eff}})$. In Eq. (7), the fourth term is the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, which is induced by the $\mathbf{E}\times\mathbf{B}$ convection of the nonadiabatic electrons, and the fifth term is the polarization drift nonlinearity, which is exactly the nonlinear term used in the Hasegawa–Mima equation. The third term is the instability drive, which introduces an effective $i\delta$ drive, $ik_y^2D_0$. An energy sink can be modeled by adding a hyperviscosity term into the model equation. This leads to a finite band of unstable drift modes with a high-k cutoff. In this paper, we focus on the drift modes in the regime $0\leqslant k\rho_s\leqslant O(1)$. Without the third and the fourth terms, that is, without the nonadiabatic electrons, the model equation reduces to the original Hasegawa–Mima equation.[1] From now on, we write $\mathbf{k}$ instead of $\mathbf{k}_\perp$ as the two-dimensional perpendicular wave vector.

In order to facilitate our analysis, we write the model equation in Fourier space as

$$i\frac{\partial}{\partial t}n_{\mathbf{k}}-\frac{\omega_{*\mathbf{k}}+ik_y^2D_0}{1+k^2\rho_s^2}n_{\mathbf{k}}+\frac{i}{1+k^2\rho_s^2}(N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}}+N_{\mathbf{k}}^{\text{POL}})=0, \quad (8)$$

where $N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}}$ is the $\mathbf{k}$th component of the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, and is written as

$$N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}} = -i\frac{1}{2} L_n D_0 \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} [(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}] \times (k_y''-k_y') n_{\mathbf{k}'} n_{\mathbf{k}''}; \tag{9}$$

the corresponding kth component of the polarization drift nonlinearity $N_{\mathbf{k}}^{\mathrm{POL}}$ is

$$N_{\mathbf{k}}^{\mathrm{POL}} = \frac{1}{2} \rho_s c_s \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} [(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}] \rho_s^2 \times (k''^2 - k'^2) n_{\mathbf{k}'} n_{\mathbf{k}''}. \tag{10}$$

Note that, in Eqs. (9) and (10), $N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}}$ and $N_{\mathbf{k}}^{\mathrm{POL}}$ are written in a form symmetric in $\mathbf{k}'$ and $\mathbf{k}''$, which follows naturally by the definition $\mathbf{k}=\mathbf{k}'+\mathbf{k}''$.

From Eq. (8), the linear dispersion relation is straightforwardly obtained by defining $i(\partial/\partial t)=\omega_{\mathbf{k}}$, and is

$$\omega_{\mathbf{k}} \equiv \omega_{\mathbf{k}}^{(0)} + i\gamma_{\mathbf{k}}^{(0)} = \frac{\omega_{*\mathbf{k}}}{1+k^2\rho_s^2} + i\frac{k_y^2 D_0}{1+k^2\rho_s^2}. \tag{11}$$

The expression for the linear growth rate $\gamma_{\mathbf{k}}^{(0)}$ justifies our previous statement that $ik_y^2 D_0$ is the instability drive.

To further understand the basic physics of this model, let us first discuss the properties of the two nonlinearities by looking at the equilibrium statistical mechanics for the model equation (7). In order to describe the equilibrium properties of the system, we assume that there are no instability drive or damping effects present in the model. In the case when there exists *only* the polarization drift nonlinearity $N^{\mathrm{POL}}=\rho_s c_s(\nabla_\perp n\times\hat{z})\cdot\nabla_\perp(\rho_s^2\nabla_\perp^2 n)$, the system has two conserved quantities. They are the total energy E and the generalized total enstrophy Ω, defined as

$$E=\frac{1}{2}\int dV(|n|^2+\rho_s^2|\nabla_\perp n|^2) = \frac{1}{2}\sum_{\mathbf{k}}(1+k^2\rho_s^2)|n_{\mathbf{k}}|^2, \tag{12}$$

$$\Omega=\frac{1}{2}\int dV(|\rho_s^2\nabla_\perp^2 n|^2+\rho_s^2|\nabla_\perp n|^2) = \frac{1}{2}\sum_{\mathbf{k}} k^2\rho_s^2(1+k^2\rho_s^2)|n_{\mathbf{k}}|^2. \tag{13}$$

The statistical mechanics prediction for the density fluctuation spectrum in an equilibrium state then is

$$\langle|n_{\mathbf{k}}|^2\rangle = \frac{1}{(1+k^2\rho_s^2)(a+bk^2\rho_s^2)}, \tag{14}$$

where a and b are Lagrangian multipliers. Thus the isotropic energy and enstrophy spectra are

$$E_k = \pi k\rho_s(1+k^2\rho_s^2)\langle|n_{\mathbf{k}}|^2\rangle = \frac{\pi k\rho_s}{a+bk^2\rho_s^2}, \tag{15}$$

$$\Omega_k = k^2\rho_s^2 E_k = \frac{\pi k^3\rho_s^3}{a+bk^2\rho_s^2}. \tag{16}$$

Note that the equilibrium energy spectrum E_k decreases at high $k\rho_s$, while the equilibrium enstrophy spectrum increases. This tendency of the system to push energy to large scales and enstrophy to small scales is the driving force behind the well-known dual cascade, of which the energy cascade is the "inverse" component. The inverse cascade of energy by the polarization drift nonlinearity was also shown dynamically by Hasegawa *et al.*[2] However, in the case when there also exists the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity $N^{\mathbf{E}\times\mathbf{B}} = -L_n D_0[\nabla_\perp(\partial n/\partial y)\times\hat{z}]\cdot\nabla_\perp n$, the system then has only one conserved quantity, that is, the total energy defined in Eq. (12). The model does not conserve the total enstrophy. Following the same analysis, the equilibrium density fluctuation spectrum is

$$\langle|n_{\mathbf{k}}|^2\rangle = \frac{c}{1+k^2\rho_s^2}, \tag{17}$$

which becomes a constant spectrum in the long-wavelength regime where $k\rho_s \ll 1$. Here, the constant c is a Lagrangian multiplier. Consequently, the isotropic energy spectrum is shown to satisfy

$$E_k = \pi c k\rho_s, \tag{18}$$

which is an increasing function of $k\rho_s$. Thus the energy is nonlinearly transferred to small scales by the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, and the long-wavelength condensation of fluctuation energy is prohibited. This is in distinct contrast to the properties of the polarization drift nonlinearity N^{POL}, and thus to the Hasegawa–Mima model.

III. CLOSURE AND NONLINEAR DISPERSION RELATION

Before we proceed with the detailed nonlinear analysis, let us first write the model equations (8)–(10) in dimensionless form by defining the ion gyroradius ρ_s as the unit of length, and the ion gyrofrequency $\Omega_i=c_s/\rho_s$ as the unit of frequency. Then the dimensionless equations are written as

$$\left(i\frac{\partial}{\partial t} - \frac{\omega_{*\mathbf{k}}+ik_y^2\tilde{D}_0}{1+k^2}\right)n_{\mathbf{k}} + \frac{i}{1+k^2}N_{\mathbf{k}}=0, \tag{19}$$

where the nonlinear term is:

$$N_{\mathbf{k}} = N_{\mathbf{k}}^{\mathbf{E}\times\mathbf{B}} + N_{\mathbf{k}}^{\mathrm{POL}} = -i\frac{1}{2}\xi\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}](k_y''-k_y')n_{\mathbf{k}'}n_{\mathbf{k}''} + \frac{1}{2}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}](k''^2-k'^2)n_{\mathbf{k}'}n_{\mathbf{k}''}. \tag{20}$$

Here, the dimensionless crossover parameter ξ (we will discuss the physical meaning of this parameter later) is defined as

$$\xi \equiv \frac{L_n D_0}{\rho_s\rho_s c_s} \equiv \alpha\sqrt{\epsilon}\left(\frac{\rho_s}{L_T}\right)\left(\frac{\Omega_i}{\nu_{\mathrm{eff}}}\right), \tag{21}$$

and the dimensionless parameter $\tilde{D}_0$ is defined as

$$\widetilde{D}_0=\xi\left(\frac{\rho_s}{L_n}\right). \quad (22)$$

In order to find the nonlinear dispersion relation for this system, we need to carry out the one-point renormalization and find the renormalized eigenvalue equation for $n_{\mathbf{k}}$. To renormalize the nonlinear equation (19), we employ the EDQNM (eddy-damped quasinormal Markovian) closure scheme,[10] which is effectively an iterative closure method, with the use of eddy damping to represent incoherent or higher-order wave correlations. Therefore, the nonlinearity can be written in terms of the driven waves, which are labeled with the superindex (2):

$$N_{\mathbf{k}}=\sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}}[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}][i\xi(k'_y+k''_y) + (k'^2-k''^2)]n_{-\mathbf{k}'}n^{(2)}_{\mathbf{k}''}, \quad (23)$$

and the driven fluctuations are the solution of

$$\left(\frac{\partial}{\partial t}+\Delta\omega_{\mathbf{k}''}\right)n^{(2)}_{\mathbf{k}''}+i\left(\frac{\omega_{*\mathbf{k}''}}{1+k''^2}+i\,\frac{k''^2_y\widetilde{D}_0}{1+k''^2}\right)n^{(2)}_{\mathbf{k}''} =\frac{(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}}{1+k''^2}[-i\xi(k'_y-k_y)+(k'^2-k^2)]n_{\mathbf{k}'}n_{\mathbf{k}}. \quad (24)$$

Here, notice that, in the above equation for $n^{(2)}_{\mathbf{k}''}$, only those direct interacting waves, i.e., the $\mathbf{k}$ and $\mathbf{k}'$ waves are kept. Thus the eddy damping rate $\Delta\omega_{\mathbf{k}''}$, which is introduced to reflect the nonlinear scrambling due to the interaction between waves other than $\mathbf{k}$ and $\mathbf{k}'$, must be determined recursively. Integrating Eq. (24), we have

$$n^{(2)}_{\mathbf{k}''}=\frac{(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}}{1+k''^2}[-i\xi(k'_y-k_y)+(k'^2-k^2)] \times\int^t dt'\exp[(-i\omega^{(0)}_{\mathbf{k}''}+\gamma^{(0)}_{\mathbf{k}''}-\Delta\omega_{\mathbf{k}''})(t-t')] \times n_{\mathbf{k}}(t')n_{\mathbf{k}'}(t'). \quad (25)$$

Further, making the ansatz for $t>t'$,

$$\langle n(t)n(t')\rangle_{\mathbf{k}}=|n_{\mathbf{k}}(0)|^2\exp[(-i\omega^{(0)}_{\mathbf{k}}+\gamma^{(0)}_{\mathbf{k}}-\Delta\omega_{\mathbf{k}}) \times(t-t')], \quad (26)$$

and substituting Eqs. (25) and (26) into Eq. (23), we finally obtain the renormalized nonlinearity

$$N_{\mathbf{k}}=\sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}}\frac{[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}]^2}{1+k''^2}R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \times\{[\xi^2(k'^2_y+k'^2_y-k^2_y)+k^2(k^2-k'^2)] +i\xi k_y(k'^2-2k^2)\}|n_{\mathbf{k}'}|^2 n_{\mathbf{k}}, \quad (27)$$

where the propagator,

$$R_{\mathbf{k},\mathbf{k}',\mathbf{k}''}=\frac{i}{(\omega^{(0)}_{\mathbf{k}}+\omega^{(0)}_{\mathbf{k}'}-\omega^{(0)}_{\mathbf{k}''})+i(\Delta\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}'}+\Delta\omega_{\mathbf{k}''}-\gamma^{(0)}_{\mathbf{k}}-\gamma^{(0)}_{\mathbf{k}'}-\gamma^{(0)}_{\mathbf{k}''})}, \quad (28)$$

represents the time scale of three-wave interactions. In this paper, we assume that the frequency mismatch is much smaller than the nonlinear propagator broadening, i.e.,

$$\omega^{(0)}_{\mathbf{k}}+\omega^{(0)}_{\mathbf{k}'}-\omega^{(0)}_{\mathbf{k}''}\ll\Delta\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}'}+\Delta\omega_{\mathbf{k}''},$$

then $R_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$ is written as

$$R_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\simeq(\Delta\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}'}+\Delta\omega_{\mathbf{k}''}-\gamma^{(0)}_{\mathbf{k}}-\gamma^{(0)}_{\mathbf{k}'}-\gamma^{(0)}_{\mathbf{k}''})^{-1}. \quad (29)$$

The above approximation can be easily justified in the long-wavelength limit, where the frequency mismatch is always zero. However, in the short-wavelength limit, this relation may be somewhat stressed. In particular, in the opposite limit where the frequency mismatch is large, the propagator $R_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\rightarrow\pi\delta(\omega^{(0)}_{\mathbf{k}}+\omega^{(0)}_{\mathbf{k}'}-\omega^{(0)}_{\mathbf{k}''})$, then the weak turbulence calculation should proceed in the usual way. In deriving the renormalized nonlinearity, we also ignore those spectral summands odd in $\mathbf{k}'$, which effectively vanish.

Now, let us discuss the renormalized nonlinearity $N_{\mathbf{k}}$ in Eq. (27). Notice that, if we had renormalized the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity (or the polarization drift nonlinearity) separately, we should have obtained only the first term (or the second term) inside the square brackets in Eq. (27). However, in the case when we have both the nonlinearities, an additional term, i.e., the last one in the curly brackets, which is the cross coupling of the two nonlinearities, appears in the renormalized nonlinearity $N_{\mathbf{k}}$. It is exactly this term that supplies a large frequency shift to the standard drift wave frequency.

Denoting $N_{\mathbf{k}}\equiv(\gamma^R_{\mathbf{k}}+i\gamma^I_{\mathbf{k}})n_{\mathbf{k}}$, the nonlinear dispersion relation is written

$$\omega_{\mathbf{k}}=\frac{\omega_{*\mathbf{k}}+\gamma^I_{\mathbf{k}}}{1+k^2}+i\,\frac{k^2_y\widetilde{D}_0-\gamma^R_{\mathbf{k}}}{1+k^2}. \quad (30)$$

Here, the real part of $N_{\mathbf{k}}$,

$$\gamma^R_{\mathbf{k}}=\sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}}\frac{[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}]^2}{1+k''^2}R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \times[\xi^2(k'^2_y+k'^2_y-k^2_y)+k^2(k^2-k'^2)]|n_{\mathbf{k}'}|^2, \quad (31)$$

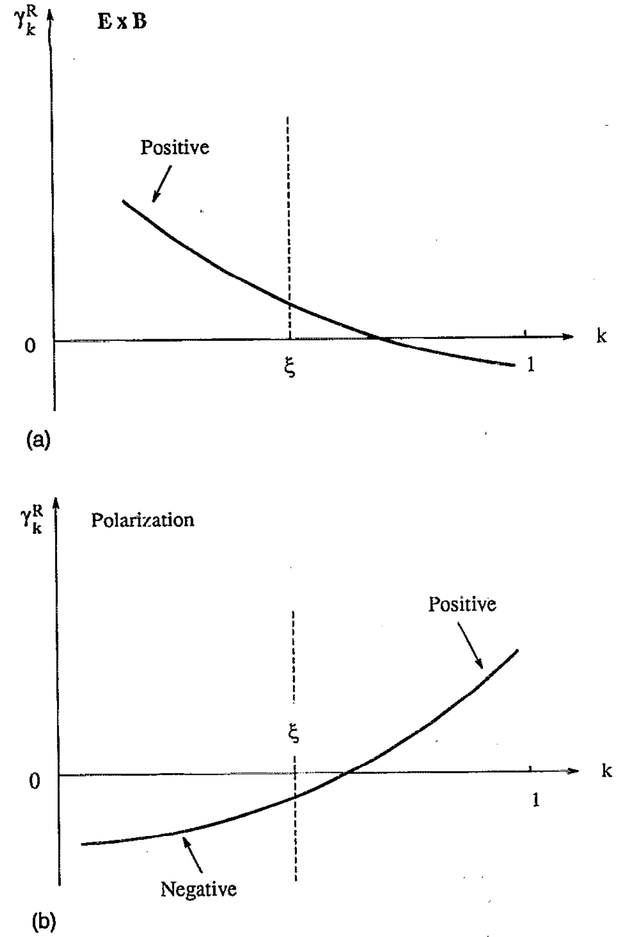

FIG. 1. Contributions to the eddy viscosity from (a) the **E**×**B** drift nonlinearity and (b) the polarization drift nonlinearity.

represents the nonlinear damping effects, i.e., the eddy viscosity. The first term of Eq. (31) comes from the **E**×**B** drift nonlinearity. Its contribution to the eddy viscosity is positive for small k, then it becomes smaller and may even become negative for large k [see Fig. 1(a)]. This trend implies that the energy transfer mediated by the **E**×**B** drift nonlinearity is from large to small scales. The second term of Eq. (31) comes from the polarization drift nonlinearity. Its contribution to the eddy viscosity, on the contrary, is negative for small k and becomes positive for large k [see Fig. 1(b)]. This trend suggests that, for the polarization drift nonlinearity, the energy transfer is from small to large scales. These observations are consistent with the analysis from the equilibrium statistical mechanics in Sec. II. Obviously, the expression for $\gamma^R_{\mathbf{k}}$ reflects the competition between the **E**×**B** and the polarization drift nonlinearities, while the comparison $2\xi^2 k'^2_y \sim k^2 k'^2$ defines the crossover wave number $k_c \sim \xi$. According to this simple comparison, the **E**×**B** drift nonlinearity dominates in the long-wavelength regime where $k<\xi$, while the polarization drift nonlinearity seems to be dominant in the short-wavelength regime where $k>\xi$. This conclusion can be also reached from dimensional analysis. As for the imaginary part of $N_{\mathbf{k}}$, i.e.,

$$\gamma^I_{\mathbf{k}}=\xi k_y \sum_{\mathbf{k}'=\mathbf{k}''-\mathbf{k}} \frac{[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}]^2}{1+k''^2} R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \times (k'^2-2k^2)\,|n_{\mathbf{k}'}|^2, \tag{32}$$

it comes from the cross coupling of the two nonlinearities, and supplies a large nonlinear frequency shift to the linear drift wave frequency. We will show in the next section, that $\gamma^I_{\mathbf{k}}$ will greatly modify the instability drive through the turbulent back reaction. Since previous studies on this subject only analyzed the two nonlinearities individually, this nonlinear frequency shift effect was overlooked. More generally, greater attention to the impact of nonlinear frequency shift effects on drift wave turbulence dynamics is clearly called for.

IV. DYNAMIC IMPACT OF THE NONLINEAR FREQUENCY SHIFT

From Eq. (30), the nonlinear frequency shift $\omega^s_{\mathbf{k}}$ is written as

$$\omega^s_{\mathbf{k}}=\xi k_y \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}]^2}{(1+k^2)(1+k''^2)} R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \times (k'^2-2k^2)\,|n_{\mathbf{k}'}|^2, \tag{33}$$

where we have changed the triad relation from $\mathbf{k}''=\mathbf{k}+\mathbf{k}'$ to $\mathbf{k}=\mathbf{k}'+\mathbf{k}''$. In order to estimate the size of $\omega^s_{\mathbf{k}}$, we define the *frequency shift parameter* $s_{\mathbf{k}}$ as

$$s_{\mathbf{k}}\equiv\frac{\omega^s_{\mathbf{k}}}{\alpha\eta_e\omega_{*\mathbf{k}}} =\frac{2}{3}\xi\frac{L_T}{\rho_s} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \frac{[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}]^2}{(1+k^2)(1+k''^2)} R_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \times (k'^2-2k^2)\,|n_{\mathbf{k}'}|^2. \tag{34}$$

We approximate $R_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$ by its mixing length value

$$R^{-1}_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \propto \xi(\rho_s/L_n)(k^2+k'^2+k''^2).$$

Thus one can see that there is a very large factor $(L_n L_T/\rho_s^2)$ in the expression for $s_{\mathbf{k}}$. Since normally, $|n_{\mathbf{k}}|^2 \propto \rho_s^2/L_n^2$, $s_{\mathbf{k}}$ is expected to be of the order of $O(\eta_e^{-1})$ (where $\eta_e>1$), in other words, $\omega^s_{\mathbf{k}}$ is expected to be comparable to $\omega_{*\mathbf{k}}$.

Since $\omega^s_{\mathbf{k}}$ approaches $\alpha\eta_e\omega_{*\mathbf{k}}$, we need to revisit the electron dynamics, and include the turbulent back reaction in the derivation of the electron density response function. By keeping this frequency shift effect, and writing Eq. (6) in dimensionless form, we have

$$\tilde{\phi}_{\mathbf{k}}=[1+ik_y\xi(1-s_{\mathbf{k}})]\tilde{n}_{\mathbf{k}}. \tag{35}$$

Once again, recall that $s_{\mathbf{k}}\sim O(\eta_e^{-1})$. At this point, we need to comment on the validity regime for the theory. In order to render the perturbative approach valid (in deriving the electrostatic potential fluctuation $\tilde{\phi}_{\mathbf{k}}$), we need to ensure that $|k_y\xi(1-s_{\mathbf{k}})|<1$, or else, a two-field model has to be used, instead. Thus, in this work, without loss of generality, we simply consider the parameter regime where the crossover parameter ξ satisfies $0<\xi<1$. Using the relation (35), and rederiving the model equation, we can easily find that the linear instability drive for mode $\mathbf{k}$ changes by a factor $(1-s_{\mathbf{k}})$, that is, in Eq. (8), the D_0 in the second term changes to $D_0(1-s_{\mathbf{k}})$. Thus the change of the linear instability drive is

$$\Delta\gamma_{\mathbf{k}}^{(0)}=-\gamma_{\mathbf{k}}^{(0)}s_{\mathbf{k}}. \tag{36}$$

This self-consistency effect will be included in the derivation of the wave kinetic equation in the next section. Here, we also need to mention that, for consistency, since the corresponding change of D_0 in the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity is of higher order, i.e., of the order $O(\eta_e^{-2})$, it is thus neglected.

Now, let us discuss the physical implications of this self-consistent frequency shift effect. First, since the change of the linear instability drive $\Delta\gamma_{\mathbf{k}}^{(0)}$ is proportional to k_y^2, we expect it to have a significant effect at short wavelength $k\sim1$, which may lead to a sort of "energy transfer" process different from that played by the polarization drift nonlinearity alone. Meanwhile, in a relative sense, the frequency shift also has a dominant effect at the crossover regime, because the other two nonlinearities compete with, and indeed almost cancel each other at the crossover point. Second, since $s_{\mathbf{k}}$ is negative at short wavelength $k\sim1$ (see Fig. 2), it enhances the instability drive there; while at long wavelength $k\ll1$, $s_{\mathbf{k}}$ becomes positive (see also Fig. 2), so it suppresses the instability drive. This observation suggests that the overall energy transfer, mediated by the nonlinear frequency shift effect, is from large to small scales. Thus this frequency shift effect, as well as the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, will also compete with the inverse transfer of energy due to the polarization drift nonlinearity.

V. WAVE KINETIC EQUATION

In this section, we derive the wave kinetic equation for the density fluctuation spectrum $|n_{\mathbf{k}}|^2$. If we multiply Eq. (19) by $n_{\mathbf{k}}^*$, substract its complex conjugate (product), and include the nonlinear frequency shift effect, the nonlinear evolution equation for the spectrum $|n_{\mathbf{k}}|^2$ is found to be

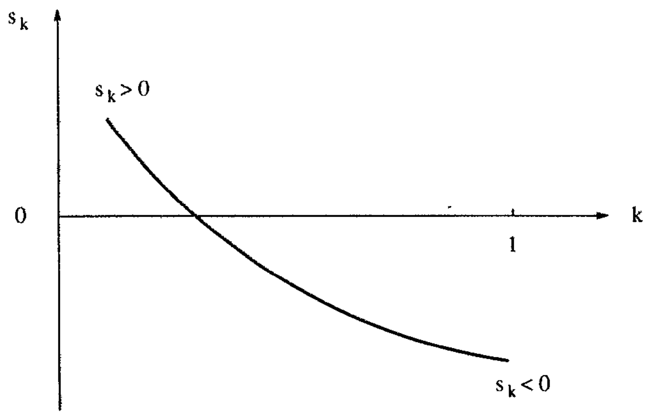

FIG. 2. $s_{\mathbf{k}}>0$ for small k, and $s_{\mathbf{k}}<0$ for large k.

$$\left(\frac{\partial}{\partial t}-2\gamma_{\mathbf{k}}^{(0)}(1-s_{\mathbf{k}})\right)|n_{\mathbf{k}}|^2+\frac{1}{1+k^2}T_{\mathbf{k}}=0, \tag{37}$$

where $T_{\mathbf{k}}$ represents the nonlinear transfer rate, and is written as

$$\begin{aligned}T_{\mathbf{k}}&\equiv N_{\mathbf{k}}n_{\mathbf{k}}^*+N_{\mathbf{k}}^*n_{\mathbf{k}}\\&=2\,\mathrm{Re}\langle N_{\mathbf{k}}n_{-\mathbf{k}}\rangle\\&=\mathrm{Re}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}][-i\xi(k_y''-k_y')\\&\quad+(k''^2-k'^2)]\langle n_{\mathbf{k}'}n_{\mathbf{k}-\mathbf{k}'}n_{-\mathbf{k}}\rangle.\end{aligned} \tag{38}$$

In order to calculate the three-body correlation function $\langle n_{\mathbf{k}'}n_{\mathbf{k}-\mathbf{k}'}n_{-\mathbf{k}}\rangle$, we again employ the EDQNM closure method. We write

$$\begin{aligned}\langle n_{\mathbf{k}'}n_{\mathbf{k}-\mathbf{k}'}n_{-\mathbf{k}}\rangle&=\langle n_{\mathbf{k}'}^{(2)}n_{\mathbf{k}-\mathbf{k}'}n_{-\mathbf{k}}\rangle+\langle n_{\mathbf{k}'}n_{\mathbf{k}-\mathbf{k}'}^{(2)}n_{-\mathbf{k}}\rangle\\&\quad+\langle n_{\mathbf{k}'}n_{\mathbf{k}-\mathbf{k}'}n_{-\mathbf{k}}^{(2)}\rangle,\end{aligned} \tag{39}$$

where the superindex (2) refers to the driven waves. Substituting Eqs. (25) and (26) into the above relation, we obtain

$$\begin{aligned}T_{\mathbf{k}}=\xi^2&\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}]^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(k_y''-k_y')\left(\frac{k_y'-k_y''}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}''}|^2+\frac{k_y''+k_y}{1+k'^2}|n_{\mathbf{k}''}|^2|n_{\mathbf{k}}|^2-\frac{k_y+k_y'}{1+k''^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2\right)\\&+\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[(\mathbf{k}\times\mathbf{k}')\cdot\hat{z}]^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(k''^2-k'^2)\left(\frac{k'^2-k''^2}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}''}|^2+\frac{k''^2-k^2}{1+k'^2}|n_{\mathbf{k}''}|^2|n_{\mathbf{k}}|^2\right.\\&\left.+\frac{k^2-k'^2}{1+k''^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2\right).\end{aligned} \tag{40}$$

Here, $\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$ is defined by noticing

$$\begin{aligned}\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}&\equiv R_{\mathbf{k},\mathbf{k}',\mathbf{k}''}=R_{\mathbf{k},-\mathbf{k}'',\mathbf{k}'}=R_{\mathbf{k},-\mathbf{k}',\mathbf{k}''}=R_{-\mathbf{k}'',-\mathbf{k}',-\mathbf{k}},\\&=[(\Delta\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}'}+\Delta\omega_{\mathbf{k}''})-\gamma_{\mathbf{k}}^{(0)}(1-s_{\mathbf{k}})-\gamma_{\mathbf{k}'}^{(0)}(1-s_{\mathbf{k}'})-\gamma_{\mathbf{k}''}^{(0)}(1-s_{\mathbf{k}''})]^{-1}.\end{aligned} \tag{41}$$

Here, again, we assume that the frequency mismatch is much smaller than the nonlinear propagator broadening, that is, we expect the system to be in a strong turbulence regime. This can be easily justified in the long-wavelength limit, where the frequency mismatch is always zero, but may be marginal in the short-wavelength limit. Notice that the first summation in the $T_{\mathbf{k}}$ equation comes from the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity (denoted as $T_{\mathbf{k}}^{\mathrm{E}\times\mathrm{B}}$), which is anisotropic in **k** space, and the second summation comes from the polarization drift nonlinearity (denoted as $T_{\mathbf{k}}^{\mathrm{POL}}$).

In a stationary state, since $(\partial/\partial t)|n_{\mathbf{k}}|^2=0$, the stationary spectrum for $|n_{\mathbf{k}}|^2$ satisfies

$$2k_y^2\tilde{D}_0(1-s_{\mathbf{k}})|n_{\mathbf{k}}|^2=T_{\mathbf{k}}^{\mathrm{E}\times\mathrm{B}}+T_{\mathbf{k}}^{\mathrm{POL}}, \tag{42}$$

where

$$T_{\mathbf{k}}^{\mathrm{E}\times\mathrm{B}}=\xi^2\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}|\mathbf{k}\times\mathbf{k}'|^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(k_y''-k_y')\times\left(\frac{k_y'-k_y''}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}''}|^2+\frac{k_y''+k_y}{1+k'^2}|n_{\mathbf{k}''}|^2|n_{\mathbf{k}}|^2-\frac{k_y+k_y'}{1+k''^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2\right), \tag{43}$$

$$T_{\mathbf{k}}^{\mathrm{POL}}=\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}|\mathbf{k}\times\mathbf{k}'|^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(k''^2-k'^2)\times\left(\frac{k'^2-k''^2}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}''}|^2+\frac{k''^2-k^2}{1+k'^2}|n_{\mathbf{k}''}|^2|n_{\mathbf{k}}|^2+\frac{k^2-k'^2}{1+k''^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2\right), \tag{44}$$

$$s_{\mathbf{k}}=\frac{2}{3}\xi\frac{L_T}{\rho_s}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\frac{|\mathbf{k}\times\mathbf{k}'|^2}{(1+k^2)(1+k''^2)}\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\times(k'^2-2k^2)|n_{\mathbf{k}'}|^2. \tag{45}$$

We will solve Eq. (42) in the next section.

VI. STATIONARY SPECTRUM

As we have mentioned in Sec. IV, in this study, we focus only on the case of which $0<\xi<1$. In this section, we will try to solve for the stationary density fluctuation spectrum in the regime $0\leqslant k\leqslant 1$. According to the dominant role played by the two nonlinearities, we define different regimes of interests as follows: (1) The $\mathbf{E}\times\mathbf{B}$ *regime* is the long-wavelength regime where the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity dominates, i.e., $0\leqslant k<\xi$; (2) The *crossover regime* is the mid-wavelength $k\sim O(\xi)$ regime where the two nonlinearities are comparable and nearly cancel each other, thus their cross-coupling effect is crucial in this regime; (3) The *Hasegawa–Mima (HM) regime*, or the *polarization regime*, is the short-wavelength regime where the polarization drift nonlinearity is thought to be dominant, i.e., $\xi<k\leqslant 1$. In the regime where $k>1$, the fluid approximation breaks down, and kinetic effects play an important role. Thus we will focus our discussion in the regime $0\leqslant k\leqslant 1$. These regimes of interest are indicated in Fig. 3.

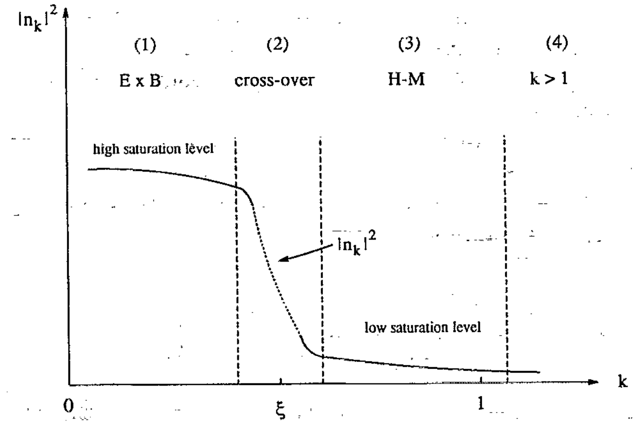

FIG. 3. Different regimes of interests and their saturation levels.

According to the simple estimate obtained by balancing the different terms in Eq. (42), the saturation level of $|n_{\mathbf{k}}|^2$ in the $\mathbf{E}\times\mathbf{B}$ regime is found to be much higher than that in the polarization regime (see also, the fluctuation level sketched in Fig. 3). We roughly can write the saturation level of $|n_{\mathbf{k}}|^2$ as $|n_{\mathbf{k}}|^2\sim[(1/k^2)(\rho_s^2/L_n^2)]$. This trend is also confirmed by computer simulations,[8,11] in which a sharp transition of the fluctuation level from the $\mathbf{E}\times\mathbf{B}$ regime to the polarization regime is observed,[11] as well. In the remaining part of this section, we will solve for the stationary spectra in different parameter ranges.

A. Spectrum in the E×B regime

In this regime, $0<k<\xi<1$. For $k<1$, compared to the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, the frequency shift nonlinearity and the polarization drift nonlinearity can be neglected. Thus the equation we are going to solve becomes

$$2k_y^2\xi\frac{\rho_s}{L_n}|n_{\mathbf{k}}|^2=\xi^2\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}|\mathbf{k}\times\mathbf{k}'|^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(-2k_y')\times\left(\frac{k_y'-k_y''}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}''}|^2+\frac{k_y''+k_y}{1+k'^2}|n_{\mathbf{k}''}|^2|n_{\mathbf{k}}|^2-\frac{k_y+k_y'}{1+k''^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2\right), \tag{46}$$

where we have used the $\mathbf{k}'$ and $\mathbf{k}''$ symmetry.

Obviously, one can tell from the k_y dependence in Eq. (46) that this is an anisotropic system. However, to simplify the problem, we view $|n_{\mathbf{k}}|^2$ as having been azimuthally averaged in **k** space (in other words, take $|n_{\mathbf{k}}|^2$ to be isotropic in **k** space). Thus $|n_{\mathbf{k}}|^2$ is approximated as a function of k^2 only. This approximation is clearly better in the Hasegawa–Mima regime than in the **E**×**B** regime. As for the propagator $\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$, it is a function of the eddy damping rate $\Delta\omega$. Thus, theoretically, the propagator is determined recursively. However, since $\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$ represents the time scale for three-wave interactions, and in this model, we have a natural time scale for mode broadening, i.e., the instability growth rate, we can approximate

$$\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \sim \frac{2}{\tilde{D}_0 k_m^2} \sim \frac{2}{\xi(\rho_s/L_n)k_m^2}, \tag{47}$$

where $k_m^2 = \max(k^2, k'^2, k''^2)$. Also, notice that all the nonlinear mode-coupling terms in Eq. (46) are products of two density fluctuation functions, and since the fluctuation level in the **E**×**B** regime is much higher than that in the polarization regime, we can neglect the contributions from the fluctuations in the polarization regime. Thus we only keep the summation until the crossover point ξ. Thus Eq. (46) is written

$$k_y^2 \frac{\rho_s^2}{L_n^2} |n_\mathbf{k}|^2 \simeq \sum_{|\mathbf{k}'|<\xi} \frac{|\mathbf{k}\times\mathbf{k}'|^2}{k_m^2} k_y'^2 [-4|n_{\mathbf{k}'}|^2 |n_{\mathbf{k}''}|^2 + 2(|n_{\mathbf{k}'}|^2 + |n_{\mathbf{k}''}|^2)|n_\mathbf{k}|^2]. \tag{48}$$

Notice here that k, k', and k'' are all in the **E**×**B** regime and are comparable, so we can use the Taylor expansion

$$|n_{\mathbf{k}'}|^2 = |n_\mathbf{k}|^2 + (k'^2 - k^2)\frac{\partial}{\partial k^2}|n_\mathbf{k}|^2, \tag{49}$$

$$|n_{\mathbf{k}-\mathbf{k}'}|^2 = |n_\mathbf{k}|^2 + (|\mathbf{k}-\mathbf{k}'|^2 - k^2)\frac{\partial}{\partial k^2}|n_\mathbf{k}|^2. \tag{50}$$

Substituting Eqs. (49) and (50) into Eq. (48), and averaging over the angle between **k** and **k**′, we have

$$\frac{\rho_s^2}{L_n^2}|n_\mathbf{k}|^2 \simeq \sum_{|\mathbf{k}'|<\xi} k'^2 \left[(k^2 - 2k'^2)|n_\mathbf{k}|^2 \frac{\partial}{\partial k^2}|n_\mathbf{k}|^2 - 2(k'^4 - k^2 k'^2)\left(\frac{\partial}{\partial k^2}|n_\mathbf{k}|^2\right)^2 \right]. \tag{51}$$

Now, we can transform the summation in Eq. (51) into integral form by defining

$$\sum_{|\mathbf{k}'|<\xi} (\Delta\mathbf{k})^2 \equiv 2\pi \int_0^\xi k'\,dk',$$

$$I(k^2) \equiv |n_\mathbf{k}|^2/(\Delta\mathbf{k})^2,$$

and

$$t \equiv k^2.$$

We finally obtain an ordinary differential equation for $I(t) \equiv I(k^2)$ as

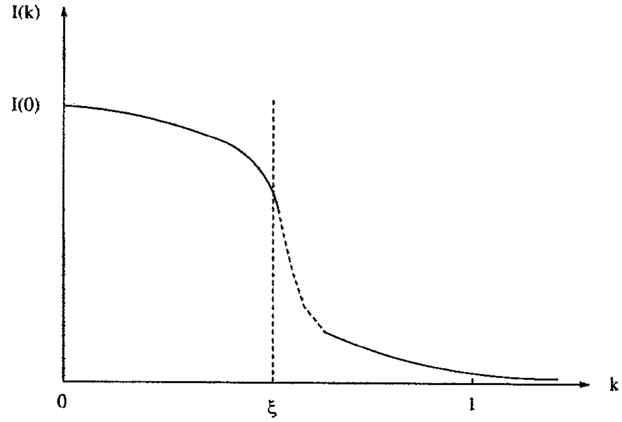

FIG. 4. Spectrum of $I(k^2)$ in different regimes.

$$\frac{\rho_s^2}{L_n^2} I(t) \simeq \pi\xi^4 \left(\frac{1}{2}t - \frac{2}{3}\xi^2\right) I(t)\frac{dI(t)}{dt} + \pi\xi^6\left(\frac{2}{3}t - \frac{1}{2}\xi^2\right) \times \left(\frac{dI(t)}{dt}\right)^2. \tag{52}$$

The second term on the right-hand side of the above equation is $O(\xi^2)$ order smaller than the first term, and thus can be neglected. Solving this ordinary differential equation, we obtain the density fluctuation spectrum in the **E**×**B** regime as

$$I(k^2) \simeq I(0) + \frac{2}{\pi\xi^4}\frac{\rho_s^2}{L_n^2} \ln\left(1 - \frac{3}{4}\frac{k^2}{\xi^2}\right). \tag{53}$$

Hence, in the $k \ll 1$ limit,

$$I(k^2) \simeq I(0) - \frac{3}{2\pi}\frac{1}{\xi^4}\frac{\rho_s^2}{L_n^2}\frac{k^2}{\xi^2}. \tag{54}$$

The spectrum in this regime is plotted in Fig. 4. From the expression in Eq. (53), one notices that the fluctuation spectrum $I(k^2)$ changes very quickly from a relatively flat density fluctuation profile to a very small fluctuation level near the crossover point $k \sim \xi$. This suggests a sharp change of the fluctuation level upon transition from the **E**×**B** regime to the polarization regime.

B. Spectrum in the Hasegawa–Mima regime

In this regime, we have $\xi < k < 1$, so the polarization drift nonlinearity is supposed to be dominant. However, in the following calculations, we will show that the leading terms in the polarization drift nonlinearity cancel exactly. Hence $N_\mathbf{k}^{\mathrm{POL}}$ actually is of the same order as $N_\mathbf{k}^{\mathbf{E}\times\mathbf{B}}$ and the frequency shift nonlinearity. This cancellation has also been observed by Hasegawa and Mima[1] and by Galeev.[12] As a consequence, *all* the nonlinearities must be retained in the spectral calculation for the Hasegawa–Mima regime. Thus we expect large modification to the Hasegawa–Mima spectrum in this regime. The equation to solve is

$$2k_y^2\xi\frac{\rho_s}{L_n}|n_{\mathbf{k}}|^2=\frac{4L_T}{3L_n}\xi^2\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}|\mathbf{k}\times\mathbf{k}'|^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\frac{k_y^2(k'^2-2k^2)}{(1+k^2)(1+k''^2)}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2+\xi^2\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}|\mathbf{k}\times\mathbf{k}'|^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(k_y''-k_y')$$
$$\times\left(\frac{k_y'-k_y''}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}''}|^2+\frac{k_y''+k_y}{1+k'^2}|n_{\mathbf{k}''}|^2|n_{\mathbf{k}}|^2-\frac{k_y+k_y'}{1+k''^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2\right)$$
$$+\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}|\mathbf{k}\times\mathbf{k}'|^2\theta_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(k''^2-k'^2)\left(\frac{k'^2-k''^2}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}''}|^2+\frac{k''^2-k^2}{1+k'^2}|n_{\mathbf{k}''}|^2|n_{\mathbf{k}}|^2\right.$$
$$\left.+\frac{k^2-k'^2}{1+k''^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2\right). \tag{55}$$

Again, we assume isotropic turbulence, i.e., $|n_{\mathbf{k}}|^2$ is a function of k^2 only. Since the saturation level $|n_{\mathbf{k}}|^2$ is much higher in the $\mathbf{E}\times\mathbf{B}$ regime than that in the polarization regime, we only consider the major nonlinear spectral transfer processes as shown in Fig. 5, that is, we only consider the triad interactions between the modes $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}-\mathbf{k}'$, with $|\mathbf{k}|\sim|\mathbf{k}-\mathbf{k}'|\sim O(1)$ in the polarization regime and $|\mathbf{k}'|\ll 1$ in the $\mathbf{E}\times\mathbf{B}$ regime in the summation over all the $\mathbf{k}'$ modes. Thus Eq. (55) can be written as

$$k_y^2\frac{\rho_s^2}{L_n^2}|n_{\mathbf{k}}|^2=-\frac{4L_T}{3L_n}\sum_{k'<\xi}\frac{|\mathbf{k}\times\mathbf{k}'|^2}{k^2}\frac{2k_y^2k^2}{(1+k^2)^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2-\sum_{k'<\xi}\frac{|\mathbf{k}\times\mathbf{k}'|^2}{k^2}\frac{2k_y^2}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}-\mathbf{k}'}|^2$$
$$-\sum_{k'<\xi}\frac{|\mathbf{k}\times\mathbf{k}'|^2}{k^2}\frac{2k_y^2}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2-\frac{1}{\xi^2}\sum_{k'<\xi}\frac{|\mathbf{k}\times\mathbf{k}'|^2}{k^2}\frac{2k^4}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}-\mathbf{k}'}|^2$$
$$+\frac{1}{\xi^2}\sum_{k'<\xi}\frac{|\mathbf{k}\times\mathbf{k}'|^2}{k^2}\frac{2k^4}{1+k^2}|n_{\mathbf{k}'}|^2|n_{\mathbf{k}}|^2. \tag{56}$$

Notice that all the contributions from both the nonlinear frequency shift effect (the first term on the right-hand side) and the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity (the second and the third term) are negative. In other words, they effectively transfer energy *nonlocally* from large to small scales. Also, notice the cancellation of the leading-order terms in the polarization drift nonlinearity (the last two terms). This cancellation is due to the fact that, for the polarization drift nonlinearity, the nonlinear self-damping (the fifth term) balances the nonlinear mode-coupling noise (the fourth term). This cancellation also suggests that the nonlinear transfer process mediated by the polarization drift nonlinearity is, to leading order, a *local* transfer. Indeed, note that, for "equilateral triads" with $|\mathbf{k}|\sim|\mathbf{k}'|\sim|\mathbf{k}-\mathbf{k}'|$, no cancellation occurs, so that local transfer persists. Thus the net effect of the polarization drift nonlinearity is comparable to the other nonlinear terms. Using the Taylor expansion in Eq. (50) (since $k'\ll 1$) and averaging over the angle between $\mathbf{k}$ and $\mathbf{k}'$, we arrive at

$$\frac{\rho_s^2}{L_n^2}|n_{\mathbf{k}}|^2+\frac{4}{3\eta_e}\frac{k^2}{(1+k^2)^2}\left(\sum_{k'<\xi}k'^2|n_{\mathbf{k}'}|^2\right)|n_{\mathbf{k}}|^2$$
$$+\frac{2}{1+k^2}\left(\sum_{k'<\xi}k'^2|n_{\mathbf{k}'}|^2\right)|n_{\mathbf{k}}|^2$$
$$+\frac{1}{\xi^2}\frac{2k^2}{1+k^2}\left(\sum_{k'<\xi}k'^4|n_{\mathbf{k}'}|^2\right)\frac{\partial}{\partial k^2}|n_{\mathbf{k}}|^2=0, \tag{57}$$

where the second term comes from the nonlinear frequency shift effect, the third term from the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, and the fourth term from the polarization drift nonlinearity. Again, by applying the transformation

$$\sum_{|\mathbf{k}'|<\xi}(\Delta\mathbf{k})^2\equiv 2\pi\int_0^\xi k'\,dk',$$
$$I(k^2)\equiv|n_{\mathbf{k}}|^2/(\Delta\mathbf{k})^2,\quad t\equiv k^2$$

and roughly approximating the fluctuation level in the $\mathbf{E}\times\mathbf{B}$ regime as a constant (since the fluctuation spectrum in that regime is fairly flat), i.e.,

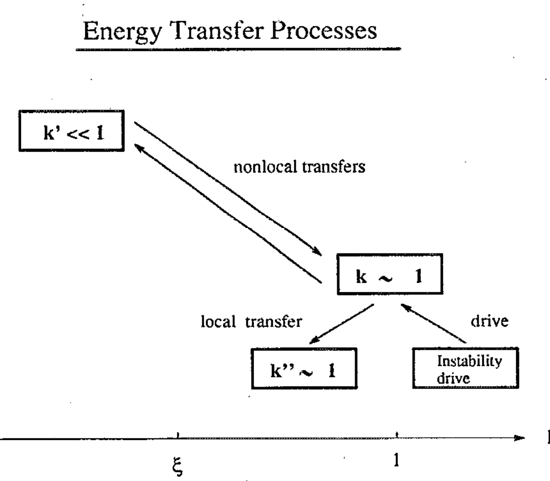

FIG. 5. Energy transfer processes (in and out of the $k\sim 1$ mode).

$$I(t')=\frac{1}{\beta\xi^4}\frac{\rho_s^2}{L_n^2},\tag{58}$$

we finally obtain an ordinary differential equation for $I(t)$ in the polarization regime as

$$\frac{dI(t)}{I(t)}=\left[-\frac{3\beta}{2\pi}-\frac{3}{2}\left(1+\frac{\beta}{\pi}\right)\frac{1}{t}-\frac{1}{\eta_e}\frac{1}{1+t}\right]dt.\tag{59}$$

Solving this ordinary differential equation, the density fluctuation spectrum in the polarization regime is

$$\frac{I(k^2)}{I(\xi^2)}=\left(\frac{k^2}{\xi^2}\right)^{-1.5(1+\beta/\pi)}\left(\frac{1+k^2}{1+\xi^2}\right)^{-1/\eta_e}\times\exp\left(-\frac{3\beta}{2\pi}(k^2-\xi^2)\right).\tag{60}$$

The spectrum in this regime is also plotted in Fig. 4. For purposes of comparison, in the case when there is no linear instability drive, the stationary spectrum has the following scaling:

$$I(k^2)\propto\frac{1}{k^3(1+k^2)^{1/\eta_e}}.\tag{61}$$

One can see that the density fluctuation spectrum in this regime is a decaying spectrum, in contrast to the Hasegawa–Mima spectrum which has a bump around $k\sim 1$.

At this point, we can see that the above results of our analysis are consistent with previous experimental observations. That is, according to the results of the far infared (FIR) laser scattering,[13] the wave-number spectra were observed to increase toward the longest measured scales. Historically, this observation was ascribed to the conventionally acknowledged inverse cascade and local transfer mechanism in wave-number space of the polarization drift nonlinearity in 2-D plasma turbulence, which is clearly inappropriate according to our analysis. On the other hand, our analysis also suggests the importance of the long-wavelength drift wave turbulence, where the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity dominates. In particular, when considering the shear flow generated by the turbulent Reynolds stress in the $L\rightarrow H$ transition, the large-scale (long-wavelength) fluctuations (where the fluctuation level is higher) will obviously play a very important role.[14]

VII. CONCLUSIONS

In this paper, we derive a one-field, two-nonlinearity model equation for dissipative drift wave turbulence. In the model, the nonlinear mode couplings by both the $\mathbf{E}\times\mathbf{B}$ drift and the polarization drift nonlinearities are present. The statistical dynamics for this dissipative drift wave turbulence is investigated using the EDQNM closure method, and the stationary density fluctuation spectrum is obtained in different spectral ranges. Here, we emphasize the following major conclusions:

(1) We show that the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity transfers energy *nonlocally* from large to small scales, and that the polarization drift nonlinearity transfers energy *locally* from small to large scales. Because of this nonlocal versus local nature of the energy transfer process, the effect of the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity is shown to be of the same order as that of the polarization drift nonlinearity in the short-wavelength regime. Thus, as we have mentioned, approximating $\delta_k=0$ in a nonlinear model for this regime is grossly invalid.

(2) A nonlinear frequency shift is induced by the cross-coupling of the $\mathbf{E}\times\mathbf{B}$ drift and polarization drift nonlinearities, and is comparable to the electron diamagnetic drift frequency at saturation. This nonlinear frequency shift effectively supplies another "nonlinearity" by self-consistently modifying the instability drive. It is shown that the energy transfer mediated by this frequency shift effect proceeds from large to small scales, as is the transfer by $\mathbf{E}\times\mathbf{B}$ drift nonlinearity.

(3) At large scales where $k_\perp\rho_s\ll 1$, it is shown that the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity dominates, so the fluctuation spectrum is determined by balancing $\mathbf{E}\times\mathbf{B}$ transfer with the linear instability drive. The spectrum is rather flat in this regime, and it decreases quickly to low levels near the crossover point ($k_\perp\rho_s\sim\xi$). The flatness of the fluctuation spectrum is due to the dominance of the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity, which has only one inviscidly conserved quantity, the energy. According to closure theory, the nonlocal transfer of energy between different drift modes leads to a flat spectrum.

(4) At small scales where $k_\perp\rho_s\sim 1$, however, *all* nonlinear transfer processes, including the nonlinear frequency shift effect, and the instability drive have to be accounted for, despite the fact that the polarization drift nonlinearity is predicted to be dominant by dimensional analysis. This leads to a *large* modification of the Hasegawa–Mima spectrum. Also, the inverse energy transfer process by the polarization drift nonlinearity alone is greatly altered by the low-k condensation blocking effect of the $\mathbf{E}\times\mathbf{B}$ drift nonlinearity. These results suggest the speculation that the dynamics of short-wavelength drift wave turbulence ($k_\perp\rho_s\sim 1$) is in fact, controlled by longer-wavelength convective cells, which are more difficult to observe.

Finally, we point out that more work is needed for this dissipative drift wave turbulence model, such work includes the nonlocal theory (i.e., in a sheared magnetic field),[15] and the velocity shear flow effects on the interaction of the two nonlinearities. These works will be presented in future publications.

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